The metrizability of L-topological groups

Fatma Bayoumi* **and Ismail Ibedou**

Department of Mathematics, Faculty of Sciences, Benha University, Benha 13518, Egypt

Abstract

This paper studies the metrizability of the notion of L-topological groups defined by Ahsanullah. We show that for any (separated) L-topological group there is an L-pseudo-metric (L-metric), in sense of Gähler which is defined using his notion of L-real numbers, compatible with the L-topology of this (separated) L-topological group. That is, any (separated) L-topological group is pseudo-metrizable (metrizable).

Keywords: Countable L-filters; Countable L-topological spaces; L-topological groups; Separated L-topological groups; L-metric spaces; L-pseudo-metric spaces; L-uniform spaces; L-filters.

1. Introduction

The notion of L-real numbers is defined and studied by S. Gähler and W. Gähler in [12]. $\mathbf{R}_L$ denotes the set of all L-real numbers. The subset $\mathbf{R}_L^*$ of $\mathbf{R}_L$ of all positive L-real numbers is used to define the L-pseudo-metric (L-metric) on a set X, by the same authors in [12], as a mapping of the cartesian product $X \times X$ to $\mathbf{R}_L^*$ which satisfies similar conditions to the conditions of the usual metric. In this paper, we study the metrizability, using the L-pseudo-metric (L-metric) in sense of [12], of a notion of L-topological group which is introduced in [1] and studied in [5]. This L-topological group is defined as a group equipped with an L-topology such that both the binary operation and the unary operation of the inverse are L-continuous with respect to this L-topology.

In this paper, using the uniformizability of L-topological groups introduced by the authors in [9], we show that any (separated) L-topological group is pseudo-metrizable (metrizable). In [9] is used the L-uniform structures which are defined in [15] on a set X, in a similar way to the usual case, as L-filters on $X \times X$.

In Section 2 of this paper we recall some results on L-filters, L-real numbers defined by Gähler in [11, 12, 13, 14], and some separation axioms defined by the authors in [2, 3, 6, 7, 8].

Sections 3 and 4 introduce and show some results on L-metric and L-uniform

*Corresponding author: e-mail: fatma_bayoumi@hotmail.com

spaces, respectively, which are needed to show the metrizability of L-topological groups.

In Section 5 we show that the L-pseudo-metric (L-metric), in sense of [12], induces the L-topology of a (separated) L-topological group, that is, any (separated) L-topological group is pseudo-metrizable (metrizable).

2. On L-filters

Here we recall some ideas concerning L-filters needed in this paper. Denote by L^X the set of all L-subsets of a non-empty set X, where L is a complete chain with different least and greatest elements 0 and 1, respectively [20]. For each L-set $\lambda \in L^X$, let λ' denote the complement of λ, defined by $\lambda'(x) = \lambda(x)'$ for all $x \in X$. For all $x \in X$ and $\alpha \in L_0$, the L-subset x_α of X whose value is α at x and 0 otherwise is called an *L-point* in X and the constant L-subset of X with value α will be denoted by $\overline{\alpha}$.

***L*-filters.** By an *L-filter* on a non-empty set X we mean [13] a mapping $\mathcal{M} : L^X \to L$ such that $\mathcal{M}(\overline{\alpha}) \leq \alpha$ for all $\alpha \in L$ and $\mathcal{M}(\overline{1}) = 1$, and also $\mathcal{M}(\lambda \wedge \mu) = \mathcal{M}(\lambda) \wedge \mathcal{M}(\mu)$ for all $\lambda, \mu \in L^X$. $\mathcal{M}$ is called *homogeneous* [11] if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$. If $\mathcal{M}$ and $\mathcal{N}$ are L-filters on X, $\mathcal{M}$ is called *finer* than $\mathcal{N}$, denoted by $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(\lambda) \geq \mathcal{N}(\lambda)$ holds for all $\lambda \in L^X$.

Let $\mathcal{F}_L X$ denote the set of all L-filters on X, $f : X \to Y$ a mapping and $\mathcal{M}$, $\mathcal{N}$ are L-filters on X, Y, respectively. Then the *image* of $\mathcal{M}$ and the *preimage* of $\mathcal{N}$ with respect to f are the L-filters $\mathcal{F}_L f(\mathcal{M})$ on Y and $\mathcal{F}_L^{-} f(\mathcal{N})$ on X defined by $\mathcal{F}_L f(\mathcal{M})(\mu) = \mathcal{M}(\mu \circ f)$ for all $\mu \in L^Y$ and $\mathcal{F}_L^{-} f(\mathcal{N})(\lambda) = \bigvee_{\mu \circ f \leq \lambda} \mathcal{N}(\mu)$ for all $\lambda \in L^X$, respectively. For each mapping $f : X \to Y$ and each L-filter $\mathcal{N}$ on Y, for which the preimage $\mathcal{F}_L^{-} f(\mathcal{N})$ exists, we have $\mathcal{F}_L f(\mathcal{F}_L^{-} f(\mathcal{N})) \leq \mathcal{N}$. Moreover, for each L-filter $\mathcal{M}$ on X, the inequality $\mathcal{M} \leq \mathcal{F}_L^{-} f(\mathcal{F}_L f(\mathcal{M}))$ holds [13].

For any set A of L-filters on X, the infimum $\bigwedge_{\mathcal{M} \in A} \mathcal{M}$, with respect to the finer relation on L-filters, does not exist in general. The infimum $\bigwedge_{\mathcal{M} \in A} \mathcal{M}$ of A exists *if and only if* for each non-empty finite subset $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of A we have $\mathcal{M}_1(\lambda_1) \wedge \cdots \wedge \mathcal{M}_n(\lambda_n) \leq \sup(\lambda_1 \wedge \cdots \wedge \lambda_n)$ for all $\lambda_1, \ldots, \lambda_n \in L^X$ [11]. If the infimum of A exists, then for each $\lambda \in L^X$ and n as a positive integer we have

$$(\bigwedge_{\mathcal{M} \in A} \mathcal{M})(\lambda) = \bigvee_{\substack{\lambda_1 \wedge \cdots \wedge \lambda_n \leq \lambda, \\ \mathcal{M}_1, \ldots, \mathcal{M}_n \in A}} (\mathcal{M}_1(\lambda_1) \wedge \cdots \wedge \mathcal{M}_n(\lambda_n)).$$

By a *filter* on X we mean a non-empty subset $\mathcal{F}$ of L^X which does not contain $\overline{0}$ and closed under finite infima and super sets [18]. For each L-filter $\mathcal{M}$ on X, the subset $\alpha\text{-pr}\,\mathcal{M}$ of L^X defined by: $\alpha\text{-pr}\,\mathcal{M} = \{\lambda \in L^X \mid \mathcal{M}(\lambda) \geq \alpha\}$ is a filter on X.

A family $(\mathcal{B}_\alpha)_{\alpha\in L_0}$ of non-empty subsets of L^X is called *valued L-filter base* on X [13] if the following conditions are fulfilled:

(V1) $\lambda \in \mathcal{B}_\alpha$ implies $\alpha \leq \sup\lambda$.

(V2) For all $\alpha, \beta \in L_0$ and all L-sets $\lambda \in \mathcal{B}_\alpha$ and $\mu \in \mathcal{B}_\beta$, if even $\alpha \wedge \beta > 0$ holds, then there are a $\gamma \geq \alpha \wedge \beta$ and an L-set $\nu \leq \lambda \wedge \mu$ such that $\nu \in \mathcal{B}_\gamma$.

Each valued L-filter base $(\mathcal{B}_\alpha)_{\alpha\in L_0}$ on a set X defines an L-filter $\mathcal{M}$ on X by: $\mathcal{M}(\lambda) = \bigvee\limits_{\mu\in\mathcal{B}_\alpha,\, \mu\leq\lambda} \alpha$ for all $\lambda \in L^X$. On the other hand, each L-filter $\mathcal{M}$ can be generated by many valued L-filter bases, and among them the greatest one $(\alpha\text{-pr}\,\mathcal{M})_{\alpha\in L_0}$.

L-neighborhood filters. In the following, in sense of [10, 16], the topology will be used and will be called L-topology. int_τ and cl_τ denote the interior and the closure operators with respect to the L-topology τ, respectively. For each L-topological space (X, τ) and each $x \in X$ the mapping $\mathcal{N}(x) : L^X \to L$ defined by: $\mathcal{N}(x)(\lambda) = \text{int}_\tau\lambda(x)$ for all $\lambda \in L^X$ is an L-filter on X, called the *L-neighborhood filter* of the space (X, τ) at x, and the mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(\lambda) = \lambda(x)$ for all $\lambda \in L^X$ is a homogeneous L-filter on X. Let (X, τ) and (Y, σ) be two L-topological spaces. Then the mapping $f : (X, \tau) \to (Y, \sigma)$ is called *L-continuous* (or *(τ, σ)-continuous*) provided $\text{int}_\sigma\mu \circ f \leq \text{int}_\tau(\mu \circ f)$ for all $\mu \in L^Y$ [14].

The L-neighborhood filter $\mathcal{N}(F)$ at an ordinary subset F of X is the L-filter on X defined, by the authors in [3], by means of $\mathcal{N}(x)$, $x \in F$ as: $\mathcal{N}(F) = \bigvee\limits_{x\in F} \mathcal{N}(x)$. The L-filter $\dot{F}$ is defined by: $\dot{F} = \bigvee\limits_{x\in F} \dot{x}$. $\dot{F} \leq \mathcal{N}(F)$ holds for all subsets F of X. Recall also here the L-filter $\dot{\lambda}$ and the L-neighborhood filter $\mathcal{N}(\lambda)$ at an L-subset λ of X which are defined by

$$\dot{\lambda} = \bigvee_{0<\lambda(x)} \dot{x} \quad \text{and} \quad \mathcal{N}(\lambda) = \bigvee_{0<\lambda(x)} \mathcal{N}(x), \tag{2.1}$$

respectively. $\dot{\lambda} \leq \mathcal{N}(\lambda)$ holds for all $\lambda \in L^X$ [4].

L-real numbers. By an L-real number is meant [12] a convex, normal, compactly supported and upper semi-continuous L-subset of the set of all real numbers $\mathbf{R}$. The set of all L-real numbers is denoted by $\mathbf{R}_L$. $\mathbf{R}$ is canonically embedded into $\mathbf{R}_L$, identifying each real number a with the crisp L-number $a^\sim$ defined by $a^\sim(\xi) = 1$ if $\xi = a$ and 0 otherwise. The set of all positive L-real numbers is defined and denoted by: $\mathbf{R}_L^* = \{x \in \mathbf{R}_L \mid x(0) = 1 \text{ and } 0^\sim \leq x\}$ and let $I_L = \{x \in \mathbf{R}_L^* \mid x \leq 1^\sim\}$, where $I = [0, 1]$ is the real unit interval. Notice that, with $\leq$ we mean that the binary operation on $\mathbf{R}_L$ defined by

$$x \leq y \Leftrightarrow x_{\alpha_1} \leq y_{\alpha_1} \text{ and } x_{\alpha_2} \leq y_{\alpha_2}$$

for all $x, y \in \mathbf{R}_L$ where $x_{\alpha_1} = \inf\{z \in \mathbf{R} \mid x(z) \geq \alpha\}$ and $x_{\alpha_2} = \sup\{z \in \mathbf{R} \mid x(z) \geq \alpha\}$ for all $x \in \mathbf{R}_L$ and for all $\alpha \in L_0$. It is shown in [13] that the class

$\{ R_\delta|_{I_L} \ \mid \ \delta \in I \} \ \cup \ \{ R^\delta|_{I_L} \ \mid \ \delta \in I \} \ \cup \ \{ 0^\sim|_{I_L} \}$ is a base for an L-topology $\Im$ on I_L, where R^δ and R_δ are the L-subsets of $\mathbf{R}_L$ defined by $R_\delta(x) = \bigvee\limits_{\alpha>\delta} x(\alpha)$ and $R^\delta(x) = (\bigvee\limits_{\alpha\geq\delta} x(\alpha))'$ for all $x \in \mathbf{R}_L$ and $\delta \in \mathbf{R}$ and note that $R_\delta|_{I_L}$, $R^\delta|_{I_L}$ are the restrictions of R_δ, R^δ on I_L, respectively. Recall also that $x \pm y$ are L-real numbers defined by $(x \pm y)(\xi) = \bigvee\limits_{\eta,\zeta\in\mathbf{R},\ \eta\pm\zeta=\xi} (x(\eta) \wedge y(\zeta))$ for all $\xi \in \mathbf{R}$. $(\mathbf{R}_L, +)$ is a commutative semi group with identity element $0^\sim$. The positive part x^+ of an L-real number x is defined as $x^+ = 0^\sim \vee x$, where

$$x - x = 0^\sim, \ (x+y)^+ \ \leq \ x^+ + y^+. \tag{2.2}$$

GT_i**-spaces.** An L-topological space (X, τ) is called [2, 6]:

(1) GT_0 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \not\leq \mathcal{N}(y)$ or $\dot{y} \not\leq \mathcal{N}(x)$.

(2) GT_1 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \not\leq \mathcal{N}(y)$ and $\dot{y} \not\leq \mathcal{N}(x)$.

(3) *completely regular* if for all $x \notin F$ and $F = \mathrm{cl}_\tau F$, there exists an L-continuous mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$.

(4) $GT_{3\frac{1}{2}}$ (or *L-Tychonoff*) if it is GT_1 and completely regular.

Proposition 2.1 [2, 3, 6, 7, 8] *Every GT_i-space is GT_{i-1}-space for all $i = 1, 2, 3, 4, 5, 6$. Moreover, the implications between GT_2-spaces, $GT_{2\frac{1}{2}}$-spaces, GT_3-spaces, $GT_{3\frac{1}{2}}$-spaces and GT_4-spaces goes as expected.*

3. Some results on L-metric spaces

A mapping $\varrho : X \times X \longrightarrow \mathbf{R}_L^*$ is called an *L-metric* [12] on X if the following conditions are fulfilled:

(1) $\varrho(x, y) = 0^\sim$ if and only if $x = y$

(2) $\varrho(x, y) = \varrho(y, x)$

(3) $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$.

If $\varrho : X \times X \longrightarrow \mathbf{R}_L^*$ satisfies the conditions (2) and (3) and the following condition:

(1)′ $\varrho(x, y) = 0^\sim$ if $x = y$

then it is called an *L-pseudo-metric* on X.

A set X equipped with an L-pseudo-metric (L-metric) ϱ on X is called an *L-pseudo-metric* (*L-metric*) *space*.

To each L-pseudo-metric (L-metric) ϱ on a set X is generated canonically a stratified L-topology τ_ϱ on X which has $\{\varepsilon \circ \varrho_x \mid \varepsilon \in \mathcal{E},\ x \in X\}$ as a base, where $\varrho_x : X \to \mathbf{R}_L^*$ is the mapping defined by $\varrho_x(y) = \varrho(x,y)$ and

$$\mathcal{E} \;=\; \{\,\overline{\alpha} \wedge R^\delta|_{\mathbf{R}_L^*} \;\mid\; \delta > 0,\ \alpha \in L\,\} \cup \{\,\overline{\alpha} \;\mid\; \alpha \in L\,\},$$

here $\overline{\alpha}$ has $\mathbf{R}_L^*$ as domain.

An L-topological space (X,τ) is called *pseudo-metrizable* (*metrizable*) if there is an L-pseudo-metric (L-metric) ϱ on X inducing τ, that is, $\tau = \tau_\varrho$.

An L-pseudo-metric ϱ is called *left* (*right*) *invariant* if

$$\varrho(x,y) = \varrho(ax,ay) \qquad (\varrho(x,y) = \varrho(xa,ya)) \text{ for all } a,x,y \in X.$$

An L-set $\lambda \in L^X$ is called *countable* (*finite*) if its support is countable (finite), where the support of λ is the set $\{x \in X \mid 0 < \lambda(x)\}$.

Let us call an L-filter $\mathcal{M}$ on a set X *countable* if the sets α-pr$\mathcal{M}$ are countable for all $\alpha \in L_0$.

Definition 3.1 An L-topological space (X,τ) is called *first countable* if every point $x \in X$ has a countable L-neighborhood filter $\mathcal{N}(x)$.

Proposition 3.1 *For any L-pseudo-metric ϱ on a set X, if τ_ϱ is the L-topology associated with ϱ, then (X,τ_ϱ) is a first countable space.*

Proof. Since $\{\varepsilon \circ \varrho_x \mid \varepsilon \in \mathcal{E},\ x \in X\}$ is a base for τ_ϱ, then for all $n \in \mathbf{N}$, the set $B_n = \{\varepsilon_n \circ \varrho_x \mid \varepsilon_n \in \mathcal{E},\ x \in X\}$, where $\varepsilon_n = \overline{\frac{1}{n}} \wedge R^\delta|_{\mathbf{R}_L^*}$, is the $\frac{1}{n}$-pr$\mathcal{N}(x)$, which implies that there exists a countable L-neighborhood filter $\mathcal{N}(x)$ at every point $x \in X$. Hence, (X,τ_ϱ) is a first countable space. □

By an *L-function family* Φ on a set X, we mean the set of all L-real functions $f : X \to I_L$.

We also have the following results.

Lemma 3.1 *Let Φ be an L-function family on X and $\sigma_f : X \times X \to I_L$ is a mapping defined by*

$$\sigma_f(x,y) \;=\; (f(x) - f(y))^+,\ f \in \Phi.$$

Then σ_f is an L-pseudo-metric on X.

Proof. Clearly, $\sigma_f(x,y) = \sigma_f(y,x)$. From (2.2), we get that $\sigma_f(x,x) = (f(x) - f(x))^+ = 0^\sim$ for all $x \in X$, and moreover

$$\sigma_f(x,y) = (f(x) - f(y))^+ \leq (f(x) - f(z))^+ \ + \ (f(z) - f(y))^+ = \sigma_f(x,z) + \sigma_f(z,y).$$

Hence, σ_f is an L-pseudo-metric on X. □

Lemma 3.2 *Let $\sigma_i : X \times X \to I_L$, $i \in I$ be an arbitrary set of L-pseudo-metrics on the set X. Then*

$$\sigma(x,y) = \sup\{\sigma_i(x,y) \mid i \in I\}$$

defines an L-pseudo-metric on X as well.

Proof. Only the triangle inequality has to be shown. For all $x, y, z \in X$ and all $i \in I$, we have

$$\sigma_i(x,y) \leq \sigma_i(x,z) + \sigma_i(z,y) \leq \sigma(x,z) + \sigma(z,y),$$

and then $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y)$. Hence, σ is an L-pseudo-metric on X. □

Here, we have shown this fact.

Lemma 3.3 *Any L-pseudo-metric ϱ on a set X is an L-metric on X* if and only if *(X, τ_ϱ) is a GT_0-space.*

Proof. Let $x, y \in X$ and $y \neq x$. Since (X, τ_ϱ) is a GT_0-space, then there exists $\mu \in L^X$ such that $\mu(x) < \beta \ \leq \ \text{int}_{\tau_\varrho}\mu(y)$ for some $\beta \in L_0$. From the definition of the base of τ_ϱ, since

$$\text{int}_{\tau_\varrho}\mu(z) = \overline{\alpha} \wedge R^\delta|_{\mathbf{R}_L^*}(\varrho(x,z)) = \alpha \wedge (\bigvee_{\eta \geq \delta} \varrho(x,z)(\eta))'$$

for all $z \in X$ and for some $\alpha \in L$, then $\varrho(x,y) = 0^\sim$ implies that $\text{int}_{\tau_\varrho}\mu(y) = \alpha \wedge 1 = \alpha$ for all $y \in X$ and all $\mu \in L^X$. Hence,

$$\alpha = \text{int}_{\tau_\varrho}\mu(x) \leq \mu(x) < \beta \leq \text{int}_{\tau_\varrho}\mu(y) = \alpha,$$

that is, $\alpha < \beta \leq \alpha$ which is a contradiction, and thus $x = y$ and ϱ is an L-metric.

Now, let $x \neq y$ and so $\varrho(x,y) \neq 0^\sim$, then there exists $\alpha > 0$ such that $\varrho(x,y)(\alpha) > 0$ and hence taking $\nu = \overline{1} \wedge R^\delta|_{\mathbf{R}_L^*} \circ \varrho_x \in L^X$, we get that

$$\nu(y) = 1 \wedge R^\delta(\varrho(x,y)) = 1 \wedge (\bigvee_{\eta \geq \delta} \varrho(x,z)(\eta))' < 1$$

whenever δ is chosen to be a very small number tends to zero. But $\text{int}_{\tau_\varrho}\nu(x) = 1 \wedge (\bigvee_{\eta \geq \delta} \varrho(x,x)(\eta))' = 1$. Hence, (X, τ_ϱ) is a GT_0-space. □

4. On L-uniform spaces

An L-filter $\mathcal{U}$ on $X \times X$ is called L-uniform structure on X [15] if the following conditions are fulfilled:

(U1) $(x,x)^{\bullet} \leq \mathcal{U}$ for all $x \in X$;

(U2) $\mathcal{U} = \mathcal{U}^{-1}$;

(U3) $\mathcal{U} \circ \mathcal{U} \leq \mathcal{U}$.

Where $(x,x)^{\bullet}(u) = u(x,x)$, $\mathcal{U}^{-1}(u) = \mathcal{U}(u^{-1})$ and $(\mathcal{U} \circ \mathcal{U})(u) = \bigvee\limits_{v \circ w \leq u} (\mathcal{U}(w) \wedge \mathcal{V}(v))$ for all $u \in L^{X \times X}$, and $u^{-1}(x,y) = u(y,x)$ and $(v \circ w)(x,y) = \bigvee\limits_{z \in X} (w(x,z) \wedge v(z,y))$ for all $x, y \in X$.

A set X equipped with an L-uniform structure $\mathcal{U}$ is called an *L-uniform space.* A mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between L-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is said to be *L-uniformly continuous* (or *$(\mathcal{U},\mathcal{V})$-continuous*) provided

$$\mathcal{F}_L(f \times f)(\mathcal{U}) \ \leq \ \mathcal{V}$$

holds. For each L-uniform structure $\mathcal{U}$ on X is associated a stratified L-topology $\tau_{\mathcal{U}}$. The related interior operator $\mathrm{int}_{\mathcal{U}}$ is given by:

$$(\mathrm{int}_{\mathcal{U}} \lambda)(x) \ = \ \mathcal{U}[\dot{x}](\lambda)$$

for all $x \in X$ and all $\lambda \in L^X$, where $\mathcal{U}[\dot{x}](\lambda) = \bigvee\limits_{u[\mu] \leq \lambda} (\mathcal{U}(u) \wedge \mu(x))$ and $u[\mu](x) = \bigvee\limits_{y \in X} (\mu(y) \wedge u(y,x))$. For all $x \in X$ and all $\lambda \in L^X$ we have

$$\mathcal{U}[\dot{x}] \ = \ \mathcal{N}(x) \ \text{ and } \ \mathcal{U}[\dot{\lambda}] = \mathcal{N}(\lambda),$$

where $\mathcal{N}(x)$ and $\mathcal{N}(\lambda)$ are the L-neighborhood filters of the space $(X, \tau_{\mathcal{U}})$ at x and λ, respectively.

Let $\mathcal{U}$ be an L-uniform structure on a set X. Then $u \in L^{X \times X}$ is called a *surrounding* provided $\mathcal{U}(u) \geq \alpha$ for some $\alpha \in L_0$ and $u = u^{-1}$. A surrounding $u \in L^{X \times X}$ is called *left (right) invariant* provided

$$u(ax, ay) = u(x,y) \qquad (u(xa, ya) = u(x,y)) \ \text{ for all } a, x, y \in X.$$

$\mathcal{U}$ is called a *left (right) invariant* L-uniform structure if $\mathcal{U}$ has a valued L-filter base consists of left (right) invariant surroundings [9].

L-topological groups. In the following we focus our study on a multiplicative group G. We denote, as usual, the identity element of G by e and the inverse of an element a of G by a^{-1}. Let G be a group and τ an L-topology on G. Then (G,τ) will be called an *L-topological group* [1, 5] if the mappings

$$\pi : (G \times G, \tau \times \tau) \to (G,\tau) \text{ defined by } \pi(a,b) = ab \text{ for all } a,b \in G$$

and

$$i : (G,\tau) \to (G,\tau) \text{ defined by } i(a) = a^{-1} \text{ for all } a \in G$$

are L-continuous. π and i are the binary operation and the unary operation of the inverse on G, respectively.

For all $\lambda \in L^G$, the inverse λ^i of λ with respect to the unary operation i on G is the L-set $\lambda \circ i$ in G defined by [9]

$$\lambda^i(x) = \lambda(x^{-1}) \text{ for all } x \in G.$$

Below, we give some examples of L-topological groups as in [5].

Example 4.1 For a group G, the induced L-topological space $(G, \omega_L(T))$ of the usual topological group (G,T) is an L-topological group.

Example 4.2 The L-real line R_L with $L = \{0,1\}$ equipped with the L- addition, defined in [20], and the L-topology on R_L is an L-topological group.

Proposition 4.1 [9] *Let (G,τ) be an L-topological group. Then there exist on G a unique left invariant L-uniform structure $\mathcal{U}^l$ and a unique right invariant L-uniform structure $\mathcal{U}^r$ compatible with τ, constructed using the family $(\alpha\text{-pr}\,\mathcal{N}(e))_{\alpha \in L_0}$ of all filters $\alpha\text{-pr}\,\mathcal{N}(e)$, where $\mathcal{N}(e)$ is the L-neighborhood filter at the identity element e of (G,τ), as follows:*

$$\mathcal{U}^l(u) = \bigvee_{v \in \mathcal{U}^l_\alpha,\, v \leq u} \alpha \qquad \text{and} \qquad \mathcal{U}^r(u) = \bigvee_{v \in \mathcal{U}^r_\alpha,\, v \leq u} \alpha, \tag{4.1}$$

where

$$\mathcal{U}^l_\alpha = \alpha\text{-pr}\,\mathcal{U}^l \qquad \text{and} \qquad \mathcal{U}^r_\alpha = \alpha\text{-pr}\,\mathcal{U}^r \tag{4.2}$$

are defined by

$$\mathcal{U}^l_\alpha = \{u \in L^{G \times G} \mid u(x,y) = (\lambda \wedge \lambda^i)(x^{-1}y) \text{ for some } \lambda \in \alpha\text{-pr}\,\mathcal{N}(e)\} \tag{4.3}$$

and

$$\mathcal{U}^r_\alpha = \{u \in L^{G \times G} \mid u(x,y) = (\lambda \wedge \lambda^i)(xy^{-1}) \text{ for some } \lambda \in \alpha\text{-pr}\,\mathcal{N}(e)\} \tag{4.4}$$

We should notice that we shall fix the notations $\mathcal{U}^l$, $\mathcal{U}^r$, $\mathcal{U}^l_\alpha$ and $\mathcal{U}^r_\alpha$ along the paper to be these defined above.

Remark 4.1 (cf. [9]) For the L-topological group (G,τ), the elements u of $\mathcal{U}^l_\alpha$ ($\mathcal{U}^r_\alpha$) are left (right) invariant surroundings. Moreover, $(\mathcal{U}^l_\alpha)_{\alpha\in L_0}$ ($(\mathcal{U}^r_\alpha)_{\alpha\in L_0}$) is a valued L-filter base for the left (right) invariant L-uniform structure $\mathcal{U}^l$ ($\mathcal{U}^r$) defined by (4.1) - (4.4), respectively.

L**-topogenous orders.** A binary relation $\ll$ on L^X is said to be an L-*topogenous order* on X [17] if the following conditions are fulfilled:

(1) $\overline{0} \ll \overline{0}$ and $\overline{1} \ll \overline{1}$;

(2) $\lambda \ll \mu$ implies $\lambda \leq \mu$;

(3) $\lambda_1 \leq \lambda \ll \mu \leq \mu_1$ implies $\lambda_1 \ll \mu_1$;

(4) From $\lambda_1 \ll \mu_1$ and $\lambda_2 \ll \mu_2$ it follows $\lambda_1 \vee \lambda_2 \ll \mu_1 \vee \mu_2$ and $\lambda_1 \wedge \lambda_2 \ll \mu_1 \wedge \mu_2$.

An L-topogenous order $\ll$ is said to be *regular* or is said to be an L-*topogenous structure* if for all $\lambda,\mu \in L^X$ with $\lambda \ll \mu$ there is a $\nu \in L^X$ such that $\lambda \ll \nu$ and $\nu \ll \mu$ hold, and $\ll$ is called *complementarily symmetric* if $\lambda \ll \mu$ implies $\mu' \ll \lambda'$ for all $\lambda,\mu \in L^X$ and moreover $\ll$ is called *perfect* if for each family $(\lambda_i)_{i\in I}$ of L-subsets of X with $\lambda_i \ll \mu$ for all $i \in I$ it follows $\bigvee_{i\in I} \lambda_i \ll \mu$.

Let $(\ll_n)$ be a sequence of L-topogenous structures on X and $(\prec_n)$ a sequence of L-topogenous structures on I_L. Then an L-real function $f : X \to I_L$ is said to be *associated with* the sequence $(\ll_n)$ if for all $\lambda,\mu \in L^{I_L}$, $\lambda \prec_n \mu$ implies $(\lambda \circ f) \ll_{n+1} (\mu \circ f)$ for every positive integer n [6].

Now, suppose that (G,τ) has a countable L-neighborhood filter $\mathcal{N}(e)$ at the identity e. Since any L-topological group, from Proposition 4.1, is uniformizable, then the left and the right invariant L-uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$, constructed also in Proposition 4.1, has, from Remark 4.1, a countable L-filter base $\mathcal{U}^l_{\underline{\frac{1}{n}}}$ and $\mathcal{U}^r_{\underline{\frac{1}{n}}}$, respectively, $n \in \mathbf{N}$.

Lemma 4.1 [4] *For all* $\lambda,\mu \in L^X$, *we have*

$$\lambda \leq \mu \ \text{ if and only if } \dot{\lambda} \leq \dot{\mu}.$$

Here, we prove this interesting result.

Lemma 4.2 *Let* $\mathcal{U}$ *be an* L*-uniform structure on a set* X, *and define a binary relation on* L^X *as follows:*

$$\lambda \ll_{\mathcal{U}} \mu \iff \mathcal{U}[\dot{\lambda}] \leq \dot{\mu}$$

for all $\lambda,\mu \in L^X$. *Then* $\ll_{\mathcal{U}}$ *is a complementarily symmetric perfect* L*-topogenous order on* X.

Proof. From the properties of $\mathcal{U}$ as an L-filter, (2.1) and Lemma 4.1 we get easily that $\ll_{\mathcal{U}}$ fulfills all the required conditions. $\square$

Proposition 4.2 [17] *There is a one - to - one correspondence between the perfect L-topogenous structures $\ll$ on a set X and the L-topologies τ on X. This correspondence is given by*

$$\lambda \ll \mu \;\Leftrightarrow\; \lambda \leq \nu \leq \mu \text{ for some } \nu \in \tau$$

for all $\lambda, \mu \in L^X$ and

$$\tau \;=\; \{\, \lambda \in L^X \mid \lambda \ll \lambda \,\}.$$

Now we have the following result.

Proposition 4.3 *Suppose that $\mathcal{U}$ and $(\mathcal{U}_{\frac{1}{n}})_{n\in N}$ are an L-uniform structure on X and its countable L-filter base, respectively, and also consider $\mathcal{V}$ an L-uniform structure on I_L. Let $(\ll_n)_{n\in \boldsymbol{N}}$ denote a sequence of complementarily symmetric perfect L-topogenous structures on X for which $\lambda \ll_n \mu \iff \mathcal{U}[\dot{\lambda}] \leq \dot{\mu}$ for all $\lambda, \mu \in L^X$, and let Φ be the family of all L-uniformly continuous functions $h : (X, \mathcal{U}) \to (I_L, \mathcal{V})$ associated with $(\ll_n)_{n\in \boldsymbol{N}}$. Then the mapping $\sigma_{\mathcal{U}} : X \times X \to I_L$ defined by*

$$\sigma_{\mathcal{U}}(x, y) \;=\; \sup\{\sigma_f(x, y) \mid f \in \Phi\},$$

where $\sigma_f(x, y) = (f(x) - f(y))^+$ for all $x, y \in X$, is an L-pseudo-metric on X and $\tau_{\mathcal{U}} = \tau_{\sigma_{\mathcal{U}}}$.

Proof. The proof of that $\sigma_{\mathcal{U}}$ is an L-pseudo-metric on X comes from Lemma 3.1, Lemma 3.2, and Lemma 4.2.

Since for any $\lambda \in L^X$, and from Proposition 4.2

$$\lambda \ll_n \lambda \iff \mathcal{U}[\dot{\lambda}] \leq \dot{\lambda}$$

means that $\lambda \in \tau_{\mathcal{U}}$ if and only if $\lambda \in \tau_{\sigma_{\mathcal{U}}}$, where $\sigma_{\mathcal{U}}$ is generated by all the L-pseudo-metrics σ_h for every h associated with $\ll_n$. Hence, $\tau_{\mathcal{U}} = \tau_{\sigma_{\mathcal{U}}}$. $\square$

5. The metrizability of L-topological groups

This section is devoted to show that any (separated) L-topological group is pseudo-metrizable (metrizable).

An L-topological group (G, τ) is called *separated* if for the identity element e, we have $\bigwedge_{\lambda\in\alpha\text{-pr}\mathcal{N}(e)} \lambda(e) \geq \alpha$, and $\bigwedge_{\lambda\in\alpha\text{-pr}\mathcal{N}(e)} \lambda(x) < \alpha$ for all $x \in G$ with $x \neq e$ and for all $\alpha \in L_0$ [9].

Proposition 5.1 [9] *Any* (separated) *L-topological group is a* ($GT3_{\frac{1}{2}}$-space) *completely regular space.*

Now, we are going to show the main result in this paper.

Proposition 5.2 *Any* (separated) *L-topological group (G, τ) is pseudo-metrizable* (metrizable).

Proof. From Proposition 4.1, we have unique left and unique right L-uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ on G defined by (4.1) such that $\tau = \tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r}$. Proposition 4.3 implies that $\tau = \tau_{\mathcal{U}^l} = \tau_{\sigma_{\mathcal{U}^l}}$ and $\tau = \tau_{\mathcal{U}^r} = \tau_{\sigma_{\mathcal{U}^r}}$, and therefore (G, τ) is pseudo-metrizable.

Also, if (G, τ) is separated, then from Proposition 5.1, we get that (G, τ) is a GT_0-space, and hence, from Lemma 3.3, we have that (G, τ) is metrizable. □

We also have the following important result.

Proposition 5.3 *Let (G, τ) be a* (separated) *L-topological group. Then the following statements are equivalent.*

(1) *τ is pseudo-metrizable* (metrizable)*;*

(2) *e has a countable L-neighborhood filter $\mathcal{N}(e)$;*

(3) *τ can be induced by a left invariant L-pseudo-metric* (L-metric)*;*

(4) *τ can be induced by a right invariant L-pseudo-metric* (L-metric).

Proof.

(1) $\Rightarrow$ (2): Follows from Proposition 3.1

(2) $\Rightarrow$ (3): Let e has a countable L-neighborhood filter $\mathcal{N}(e)$, and let $\mathcal{U}^l_{\frac{1}{n}}$ be a countable L-filter base of the left invariant L-uniform structure $\mathcal{U}^l$, defined by (4.1), which is compatible with τ. Then, from Lemma 4.2, $\lambda \ll_{\mathcal{U}^l} \mu \;\Leftrightarrow\; \mathcal{U}^l[\dot{\lambda}] \leq \dot{\mu}$ for all $\lambda, \mu \in L^G$ defines a sequence of complementarily symmetric perfect L-topogenous structures on G. Taking $\mathcal{V}$ as an L-uniform structure on I_L and Φ as the family of all L-uniformly continuous functions $h : (G, \mathcal{U}^l) \to (I_L, \mathcal{V})$ associated with $\ll_{\mathcal{U}^l}$, we get, from Proposition 4.3, that the L-mapping $\sigma : G \times G \to I_L$ defined by $\sigma(x, y) = \sup\{(f(x) - f(y))^+ \mid f \in \Phi\}$ is an L-pseudo-metric on G and $\tau = \tau_{\mathcal{U}^l} = \tau_{\sigma_{\mathcal{U}^l}}$.

Now, we define $h_a : G \to I_L$ by $h_a(x) = h(ax)$ for all $a, x \in G$. From $h \in \Phi$ is L-uniformly continuous, that is, $\mathcal{F}_L(h \times h)(\mathcal{U}^l) \leq \mathcal{V}$ and that the elements of $\mathcal{U}^l_{\frac{1}{n}}$ are left invariant from Remark 4.1, and from (4.1), we have

$$\begin{array}{rcl}\mathcal{F}_L(h_a \times h_a)\mathcal{U}^l(v) & = & \mathcal{U}^l(v \circ (h_a \times h_a)) \\ & = & \bigvee\limits_{u \in \mathcal{U}^l_{\frac{1}{n}},\, u \leq v \circ (h_a \times h_a)} \alpha \\ & = & \bigvee\limits_{u \in \mathcal{U}^l_{\frac{1}{n}},\, u \leq v \circ (h \times h)} \alpha \\ & = & \mathcal{F}_L(h \times h)\mathcal{U}^l(v) \\ & \geq & \mathcal{V}(v).\end{array}$$

Hence, h_a is L-uniformly continuous associated with $\ll_{\mathcal{U}^l}$, that is, $h_a \in \Phi$. Thus

$$\begin{array}{rcl}\sigma(ax, ay) & = & \sup\{(h(ax) - h(ay))^+ \mid h \in \Phi\} \\ & = & \sup\{(h_a(x) - h_a(y))^+ \mid h \in \Phi\} \\ & \leq & \sup\{(k(x) - k(y))^+ \mid k \in \Phi\} \\ & = & \sigma(x, y).\end{array}$$

Applying the same for a^{-1} instead of a, we get that

$$\sigma(x, y) = \sigma(a^{-1}ax, a^{-1}ay) \leq \sigma(ax, ay).$$

That is, $\sigma(ax, ay) = \sigma(x, y)$ for all $a, x, y \in G$ and then σ is a left invariant L-pseudo-metric on G inducing τ.

(2) $\Rightarrow$ (4): By a similar proof as in the case (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1): Obvious.

The proposition is still true if we consider the parentheses. □

Example 5.1 From Proposition 5.2, we can deduce that any L-topological group (G, τ) on which there can be constructed an L-uniform structure $\mathcal{U}$ compatible with τ is pseudo-metrizable in general and is metrizable whenever (G, τ) is separated.

References

[1] T. M. G. Ahsanullah; *On fuzzy neighborhood groups*, J. Math. Anal. Appl. 130 (1988) 237 - 251.

[2] F. Bayoumi, I. Ibedou; *T_i-spaces, I*, The Journal of The Egyptian Mathematical Society 10 (2002) 179 - 199.

[3] F. Bayoumi, I. Ibedou; *T_i-spaces, II*, The Journal of The Egyptian Mathematical Society 10 (2002) 201 - 215.

[4] F. Bayoumi, I. Ibedou; *The relation between the GT_i-spaces and fuzzy proximity spaces, G-compact spaces, fuzzy uniform spaces*, The Journal of Chaos, Solitons and Fractals 20 (2004) 955 - 966.

[5] F. Bayoumi; *On initial and final L-topological groups*, Fuzzy Sets and Systems 156 (2005) 43 - 54.

[6] F. Bayoumi, I. Ibedou; *$GT_{3\frac{1}{2}}$-spaces, I*, The Journal of the Egyptian Mathematical Society, Accepted for publication November 1, 2005.

[7] F. Bayoumi, I. Ibedou; *$GT_{3\frac{1}{2}}$-spaces, II*, The Journal of the Egyptian Mathematical Society, Accepted for publication November 1, 2005.

[8] F. Bayoumi, I. Ibedou; *$GT_{2\frac{1}{2}}$-spaces, GT_5-spaces and GT_6-spaces*, submitted.

[9] F. Bayoumi, I. Ibedou; *The uniformizability of L-topological groups*, submitted.

[10] C. H. Chang; *Fuzzy topological spaces*, J. Math. Anal. Appl. 24 (1968) 182 - 190.

[11] P. Eklund, W. Gähler; *Fuzzy filter functors and convergence*, in: S. E. Rodabaugh, E. P. Klement, U.Höhle; *Applications of Category Theory to Fuzzy Subsets*, Kluwer Academic Publishers, (1992) 109 - 136.

[12] S. Gähler, W. Gähler; *Fuzzy real numbers*, Fuzzy Sets and Systems 66 (1994) 137 - 158.

[13] W. Gähler; *The general fuzzy filter approach to fuzzy topology, I*, Fuzzy Sets and Systems 76 (1995) 205 - 224.

[14] W. Gähler; *The general fuzzy filter approach to fuzzy topology, II*, Fuzzy Sets and Systems 76 (1995) 225-246.

[15] W. Gähler, F. Bayoumi, A. Kandil, A. Nouh; *The theory of global fuzzy neighborhood structures, III, Fuzzy uniform structures*, Fuzzy Sets and Systems 98 (1998) 175 - 199.

[16] J. A. Goguen; *L-fuzzy sets*, J. Math. Anal. Appl. 18 (1967) 145 - 174.

[17] A. K. Katsaras, C. G. Petalas; *On fuzzy syntopogenous structures*, J. Math. Anal. Appl. 99 (1984) 219 - 236.

[18] R. Lowen; *Convergence in fuzzy topological spaces*, General Topol. Appl. 10 (1979) 147 - 160.

[19] S. E. Rodabaugh, E. P. Klement, U.Höhle; *Applications of Category Theory to Fuzzy Subsets*, Kluwer Academic Publishers, (1992) 109 - 136.

[20] L. A. Zadeh; *Fuzzy sets*, Information and Control 8 (1965) 338 - 353.